Test_2 - sequences & series, binomial theorem  

11 questions. GDC is allowed on all questions.  
total marks on test: 65

1. Consider the following series: $-13 - 5 + 3 + \cdots + 83$
   
   (a) Express the series in sigma notation.  
   
   (b) Find the sum of the series.  

   \[ a_1 = -13, \quad d = 8 \]  
   \[ u_n = u_1 + (n-1)d \]  
   \[ 83 = -13 + (n-1)8 \]  
   \[ 96 = 8n - 8 \]  
   \[ 8n = 104 \]  
   \[ n = 13 \]  
   
   rule for $n$th term: \[ u_n = -13 + (n-1)8 = -13 + 8n - 8 = 8n - 21 \]  
   
   thus, \[ -13 - 5 + 3 + \cdots + 83 = \sum_{n=1}^{13} (8n - 21) \]

   (b) \[ S_n = \frac{n}{2} (u_1 + u_n) \]  
   \[ S_{13} = \frac{13}{2} (-13 + 83) \]  
   \[ S_{13} = 455 \]

2. A geometric series has a negative common ratio. The sum of the first two terms is 6, and the sum to infinity is 8. Find the common ratio and the first two terms.  

   \[ u_1 + u_1r = 6 \]  
   \[ u_1 (1+r) = 6 \]  
   \[ u_1 = \frac{6}{1+r} \]  

   \[ S_\infty = \frac{u_1}{1-r} = 8 \]  
   \[ \frac{6}{1+r} = 8 \]  
   \[ \frac{6}{1-r} = 8 \]  
   \[ (1+r)(1-r) \]

   \[ 1-r^2 = \frac{3}{4} \]  
   \[ r^2 = \frac{1}{4} \]  
   \[ r = \pm \frac{1}{2} \]  

   negative common ratio  
   \[ r = -\frac{1}{2} \]

   \[ u_1 = \frac{6}{1-\frac{1}{2}} = 12 \]  
   \[ u_2 = 12 \left(-\frac{1}{2}\right) = -6 \]  

   \[ r = -\frac{1}{2}, \quad u_1 = 12, \quad u_2 = -6 \]
3. An arithmetic sequence with 9 terms has a common difference of $-8$ and a sum of 45. Find the value of the first term and the value of the last term. [6 marks]

\[
S_9 = \frac{9}{2} (u_1 + u_9) = 45 \Rightarrow u_1 + u_9 = 10
\]

\[
u_9 = u_1 + (9-1)(-8) \Rightarrow u_1 - u_9 = 64
\]

\[
2u_1 = 74 \Rightarrow u_1 = 37
\]

\[
u_9 = 10 - u_1 \Rightarrow u_9 = -27
\]

First term $= 37$, last term $= -27$.

4. Find the sum of all the multiples of 3 between 100 and 800. [5 marks]

\[
798 = 3 \cdot 266
\]

\[
u_1 = 102, \ u_n = 798, \ d = 3
\]

\[
u_n = u_1 + (n-1)d \Rightarrow 798 = 102 + (n-1)3 \Rightarrow 696 = 3n - 3
\]

\[
3n = 699 \Rightarrow n = 233
\]

\[
S_{233} = \frac{233}{2} (102 + 798)
\]

\[
S_{233} = 104850
\]
5. The sum to infinity of a geometric series is \( \frac{9}{4} \). The sum of the first four terms is \( \frac{20}{9} \). Find the first term, \( u_1 \), and the common ratio, \( r \). [7 marks]

\[
S_\infty = \frac{u_1}{1-r} = \frac{9}{4} \quad \Rightarrow \quad u_1 = \frac{9}{4} (1-r)
\]

\[
u_1 + u_1r + u_1r^2 + u_1r^3 = \frac{20}{9} \quad \Rightarrow \quad \frac{9}{4} (1-r) + \frac{9}{4} (1-r) r + \frac{9}{4} (1-r) r^2 + \frac{9}{4} (1-r) r^3 = \frac{20}{9}
\]

Multiply both sides by \( \frac{4}{9} \)

\[
1-r + (1-r)r + (1-r)r^2 + (1-r)r^3 = \frac{80}{81}
\]

\[
1-r - r^2 + r^3 + r^4 - \frac{80}{81} = 0
\]

\[
1-r^4 = \frac{80}{81} \quad \Rightarrow \quad r^4 = \frac{81}{81} \quad \Rightarrow \quad r = \pm \frac{1}{3}
\]

If \( r = \frac{1}{3} \), \( u_1 = \frac{9}{4} (1-\frac{1}{3}) = \frac{3}{2} \)

If \( r = -\frac{1}{3} \), \( u_1 = \frac{9}{4} (1+\frac{1}{3}) = 3 \)

\[
r = \frac{1}{3}, \quad u_1 = \frac{3}{2} \quad \text{OR} \quad r = -\frac{1}{3}, \quad u_1 = 3
\]

6. Evaluate: \( \sum_{r=0}^{7} 32 \left( \frac{3}{2} \right)^r \) [5 marks]

\[
u_1 = 32 \left( \frac{3}{2} \right)^0 = 32
\]

\[
\sum_n = \frac{u_1 (1-r^n)}{1-r}
\]

\[
S_8 = \frac{32 \left( 1-\left( \frac{3}{2} \right)^8 \right)}{1-\frac{3}{2}}
\]

\[
S_8 = \frac{6305}{4} = 1576.25
\]
7. Consider the infinite geometric series \( 1 + \frac{2x}{3} + \left(\frac{2x}{3}\right)^2 + \left(\frac{2x}{3}\right)^3 + \ldots \).

(a) For what values of \( x \) does the series converge? [4 marks]

(b) Find the sum of the series if \( x = 1.2 \) [2 marks]

\[ r = \frac{2x}{3} \]

Solve:
\[-1 < \frac{2x}{3} < 1\]
\[
\frac{2x}{3} > -1 \quad \text{AND} \quad \frac{2x}{3} < 1
\]
\[ 2x > -3 \quad \text{AND} \quad 2x < 3 \]
\[ x > -\frac{3}{2} \quad \text{AND} \quad x < \frac{3}{2} \]
\[ -\frac{3}{2} < x < \frac{3}{2} \]

\( S_{\infty} = \frac{u_1}{1-r} = \frac{1}{1 - \frac{2(1,2)}{3}} \)

\( S_{\infty} = 5 \)

8. Find the coefficient of the \( x^3 \) term in the expansion of \( \left(2 - \frac{3x}{2}\right)^6 \). [6 marks]

General term:
\[ \binom{6}{r} (2)^{6-r} \left(-\frac{3x}{2}\right)^r \]

\( r = 3 \) for \( x^3 \) term

Substituting \( r = 3 \):
\[ \binom{6}{3} (2)^3 \left(-\frac{3x}{2}\right)^3 = 20 \cdot 8 \left(-\frac{27x^3}{8}\right) = -540x^3 \]

Coefficient of \( x^3 \) term is \( -540 \)
9. Find the coefficient of $x^2$ in the expansion of $(x+1)^2 (2x-1)^4$. 

$$(x+1)^2 (2x-1)^4 = (x^2 + 2x + 1) \left( \ldots \left( \frac{4}{1!} (2x)^2 (-1)^2 + \frac{4}{3!} (2x)^1 (-1)^3 + \frac{4}{5!} (2x)^0 (-1)^4 \right) \right)$$

$$= (x^2 + 2x + 1) \left( \ldots \ 24x^2 - 8x + 1 \right)$$

Computing only $x^2$ terms: $x^2 \cdot 1 + 2x(-8x) + 1 \cdot 24x^2 = 9x^2$

Coefficient of $x^2$ is 9

10. Find the term independent of $x$ in the expansion of $\left( \frac{1}{x^2} - 2x \right)^6$.

General term: $\binom{6}{r} \left( \frac{1}{x^2} \right)^{6-r} (-2x)^r = \binom{6}{r} (x^{-2})^{6-r} (-2x)^r$

Find $r$ that produces exponent of zero: $-2(6-r) + r = 0 \Rightarrow r = 4$

Substituting $r=4$: $\binom{6}{4} (x^{-2})^2 (-2x)^4 = 15 (x^{-2}) (16x^4) = 240$

Constant term is 240
11. The coefficient of $x$ in the expansion \( \left( x + \frac{1}{ax^2} \right)^7 \) is \( \frac{7}{3} \). Find the possible values of $a$.  

[6 marks]

\[ \text{general term: } \binom{7}{r} \left(x\right)^{7-r} \left(\frac{x^{-2}}{a}\right)^r \]

Find \( r \) that produces exponent of one: \( 7 - r - 2r = 1 \) \( \Rightarrow r = 2 \)

\[ \text{x term, substituting } r = 2: \binom{7}{2} \left(\frac{x^{-2}}{a}\right)^2 = \frac{7}{3} x \]

\[ 21 x^5 \left(\frac{x^{-4}}{a^2}\right) = \frac{7}{3} x \]

\[ \frac{21}{a^2} = \frac{7}{3} \Rightarrow a^2 = 9 \Rightarrow a = \pm 3 \]

**Bonus question:** Find $n$ if the coefficient of $x^2$ in the expansion of \( (1 + 2x)^n \) is 112.  

[+2 marks]

\[ \text{x}^2 \text{ term: } \binom{n}{2} (1)^{n-2} (2x)^2 = \binom{n}{2} 4x^2 = 112 x^2 \]

\[ 4 \binom{n}{2} = 112 \Rightarrow \binom{n}{2} = 28 \]

\[ \frac{n!}{2! (n-2)!} = 28 \Rightarrow \frac{n!}{(n-2)!} = 56 \]

\[ n (n-1) = 56 \Rightarrow n^2 - n - 56 = 0 \]

\[ (n-8)(n+7) = 5 \]

\[ n = 8 \text{ or } n = -7 \]

\[ \text{Not possible} \]

\[ n = 8 \]