Transformations of Graphs

\[ f(x) \Rightarrow g(x) = a \cdot \left[b(x+c)\right]+d \]

- translations: \( y = f(x) + d; \quad y = f(x+c) \)
- reflections (in both axes): \( y = -f(x); \quad y = f(-x) \)
- vertical stretch with scale factor \( a \): \( y = a \cdot f(x) \)
- horizontal stretch with scale factor \( \frac{1}{b} \): \( y = f(bx) \)
- composite transformations

[answers on next page]

Exercises

A calculator may be used to visually check answers.

1. For each of the following, describe fully the transformation(s) which maps the graph of \( f(x) \) onto the graph of \( g(x) \).
   
   (a) \( f(x) = 4x; \quad g(x) = 4x-3 \)
   
   (b) \( f(x) = (x+1)^2; \quad g(x) = (x+4)^2 \)
   
   (c) \( f(x) = x^3 + 2x - 3; \quad g(x) = x^3 + 2x + 5 \)
   
   (d) \( f(x) = e^x; \quad g(x) = e^{-x} \)
   
   (e) \( f(x) = x - \frac{1}{x}; \quad g(x) = \frac{1}{x} - x \)
   
   (f) \( f(x) = \log x; \quad g(x) = 2 + \log(-x) \)
   
   (g) \( f(x) = \sin x; \quad g(x) = -3\sin\left(x + \frac{\pi}{4}\right) \)

2. (a) Use the technique of completing the square to write the expression \( x^2 - 6x + 11 \) in the form \( (x+c)^2 + d \).
   
   (b) Thus, describe fully the transformations that map the graph of \( y = x^2 \) onto the graph of \( y = x^2 - 6x + 11 \).

3. Let \( f(x) = 2\sqrt{x+3} + 5 \) and \( h(x) = \sqrt{x} \). The graph of \( f \) may be obtained from the graph of \( h \) by the following two transformations: a stretch of scale factor \( k \) in the \( y \)-direction followed by a translation of \( \left(\frac{p}{q}\right) \). Write down \( \left(\frac{p}{q}\right) \) and the value of \( k \).
Transformations of Graphs

4. Consider the quadratic function \( g(x) = 3x^2 + 12x + 8 \)

   (a) Write \( g \) in the form \( g(x) = 3(x+h)^2 + k \).

   (b) The graph of \( g \) is translated 4 units in the negative \( x \)-direction and 6 units in the positive \( y \)-direction. Find the function \( g \) for the translated graph, giving your answer in the form \( g(x) = 3(x+c)^2 + d \).

5. Let \( f(x) = \frac{1}{x-2}, \; x \neq 2 \).

   (a) Sketch the graph of \( f \), clearly indicating any asymptotes or any \( x \)- or \( y \)-intercepts.

   (b) The graph of function \( g \) is obtained by first reflecting the graph of \( f \) about the \( x \)-axis followed by a translation 3 units in the positive \( y \)-direction. Write an expression for the function \( g \) and sketch its graph, clearly indicating any asymptotes or any \( x \)- or \( y \)-intercepts.

   (c) The graph of function \( h \) is obtained by first translating the graph of \( f \) 3 units in the positive \( y \)-direction followed by a reflection about the \( x \)-axis. Write an expression for the function \( h \) and sketch its graph, clearly indicating any asymptotes or any \( x \)- or \( y \)-intercepts.

   (d) Comparing the results for parts (b) and (c), comment on whether it makes a difference in the order in which the transformations of a reflection about the \( x \)-axis and a vertical translation are applied.

6. Part of the graph of the function \( y = f(x) \) and its vertical and horizontal asymptotes are shown below. On a separate pair of coordinate axes, sketch the graph of \( y = -f(x+1) \). Clearly indicate any asymptotes on the graph along with writing the equation of the asymptote.

![Graph of a function with asymptotes](image)

7. The graph of the function \( y = 2x^2 - 3 \) is shrunk horizontally by a factor of \( \frac{1}{2} \) and then translated by \( \left( \begin{array}{c} 1 \\ -2 \end{array} \right) \). Find the equation of the resulting graph expressing it in the form \( y = ax^2 + bx + c \).
Transformations of Graphs

ANSWERS

1. (a) vertical translation down 3 units maps graph of \( f(x) \) onto the graph of \( g(x) \)
   (b) horizontal translation left 3 units maps graph of \( f(x) \) onto the graph of \( g(x) \)
   (c) vertical translation up 8 units maps graph of \( f(x) \) onto the graph of \( g(x) \)
   (d) a reflection about the \( y \)-axis maps graph of \( f(x) \) onto the graph of \( g(x) \)
   (e) a reflection about the \( x \)-axis maps graph of \( f(x) \) onto the graph of \( g(x) \)
   (f) a reflection about the \( y \)-axis followed by vertical translation up 2 units maps \( f(x) \) onto \( g(x) \); OR a vertical translation up 2 units followed by a reflection about the \( y \)-axis
   (g) graph of \( f(x) \) mapped onto graph of \( g(x) \) by the three transformations: reflection about the \( x \)-axis; vertical stretch by a factor of 3; horizontal translation left \( \frac{\pi}{4} \) units (in any order)

2. (a) \( x^2 - 6x + 11 = (x - 3)^2 + 2 \)
   (b) horizontal translation 3 units right and vertical translation 2 units up (in either order)

3. \( k = 2; \begin{pmatrix} -3 \\ 5 \end{pmatrix} \)

4. (a) \( g(x) = 3(x + 2)^2 - 4 \)
   (b) \( g(x) = 3(x + 6)^2 + 2 \)

5. (a) [Diagram of a graph with asymptotes and labels for horizontal and vertical asymptotes]
5. (continued)

(b) \( g(x) = -\frac{1}{x-2} + 3, \ x \neq 2 \)

\[
\begin{align*}
\text{Vertical asymptote} \ x &= 2 \\
\text{Horizontal asymptote} \ y &= 3 \\
(0, \frac{7}{2})
\end{align*}
\]

(c) \( h(x) = -\frac{1}{x-2} - 3, \ x \neq 2 \)

\[
\begin{align*}
\text{Vertical asymptote} \ x &= 2 \\
\text{Horizontal asymptote} \ y &= -3 \\
(0, -\frac{7}{2}) \\
(\frac{3}{2}, 0)
\end{align*}
\]

(d) The functions obtained in (b) and (c) are different when the order in which the transformations are applied is switched. The order in which the transformations of a reflection about the \( x \)-axis and a vertical translation are applied to a function does make a difference.

6.

7. \( y = 8x^2 - 16x + 3 \)